ELEN E3401: Electromagnetics

Spring 2025

Prof. Keren Bergman

Lecture #14

Course Project

Part 1: DUE Friday March 28: Team Formation and Topic (10%)
Submit the names of the team members and the project topic.

Each team should have three students. Please choose a project topic from the list or create your own. If you decide to choose a topic not on the list, email your suggestion to the TAs (Brian, (zw2542@columbia.edu) and Brett (bcg2133@columbia.edu), and Professor Bergman for approval by noon Thursday March 27.

1. Fiber Optics
2. Medical Imaging
3. OLEDs
4. RF Antennas
5. LiDAR
6. Optical Metrology
7. Integrated Photonics
8. Solar Cells
9. Imaging Sensors
10. Free space communications
11. Photonic Computing

Course Project

Part 2: DUE Monday April 7: Project Proposal (20%)

Each team submits a project proposal with a description of the topic and equally assigned components of the EM subarea for each of the members. This outline should include a half page description for each team member's tasks as well as any references (books, articles, relevant websites, etc.).

Part 3: Monday May 5: Final Project Presentations (40%)

Each team will have a slot for presentation on their projects, each member must speak for 5min. An additional 3min at the beginning of the presentation should be dedicated to the introduction of the topic and sub-topics. An additional 3min at the end will be used for questions and discussion.

Part 4: DUE Friday May 9 (30%)

Submit paper on the specific task, including quantitative analysis. The papers will be prepared and submitted individually by every student, however team discussion is appropriate. Papers are limited to 6 pages inclusive of figures (references are not counted in the 6 pages).

Magnetostatics

Gauss' Law $\vec{\nabla} \cdot \vec{D} = \rho_V$

$\vec{\nabla} \cdot \vec{B} = 0$ Gauss' Law (mag)

Faraday's Law $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$ Ampere's Law

<u>Magnetostatics:</u>

$$\frac{\partial}{\partial t} = 0$$

$$\boxed{\begin{gathered} \vec{\nabla} \cdot \vec{B} = 0 \\ \vec{\nabla} \times \vec{H} = \vec{J} \end{gathered}}$$

$\vec{B}$: Magnetic flux density

$\vec{H}$: Magnetic field intensity

$$\vec{B} = \mu \vec{H}$$

For most materials, $\mu = \mu_0$
(except ferromagnets)

Integral form:

$$\oint_S \vec{B} \cdot d\vec{s} = 0 \qquad \oint_C \vec{H} \cdot d\vec{l} = I$$

Magnetic force

We defined $\vec{E}$ through electric force on charge: $\vec{F}_e = q\vec{E}$

Define magnetic flux density, $\overrightarrow{B}$:

in terms of force $\vec{F}_m$ acting <u>on moving charge q</u> with velocity = $\vec{u}$:

$$\vec{F}_m = q\vec{u} \times \vec{B} \qquad [\mathrm{N}]$$

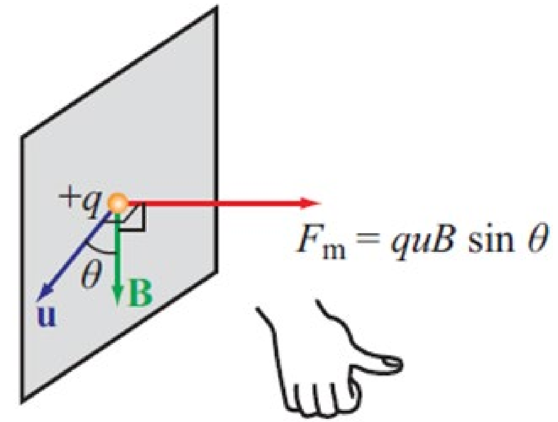
Magnetic flux density, $\vec{B}$ units:
N/C ·m/s →Tesla

Lorenz force

For + charge, direction of $\vec{F}_m$ is $\vec{u} \times \vec{B}$

$$\left|\vec{F}_m\right| = quBsin\theta$$

F_m direction depends on charge polarity and velocity



(a)

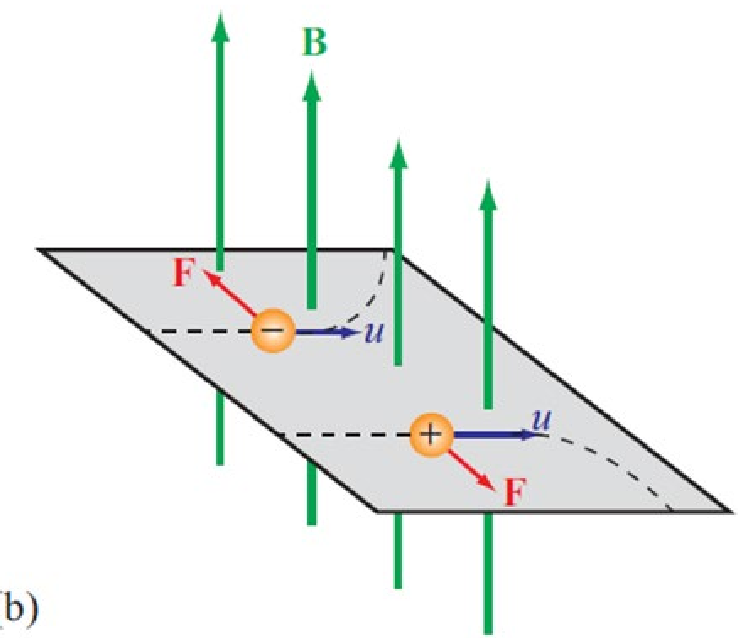
Charge with both $\vec{E}$ and $\vec{B}$:

$$\vec{F} = \vec{F}_e + \vec{F}_m = q\vec{E} + q\vec{u} \times \vec{B}$$

$$= q(\underbrace{\vec{E}}_{\text{In same direction as } \vec{E}} + \underbrace{\vec{u} \times \vec{B}}_{\perp \text{ to } \vec{B}\text{; acts only on moving charge}})$$

In same direction as $\vec{E}$

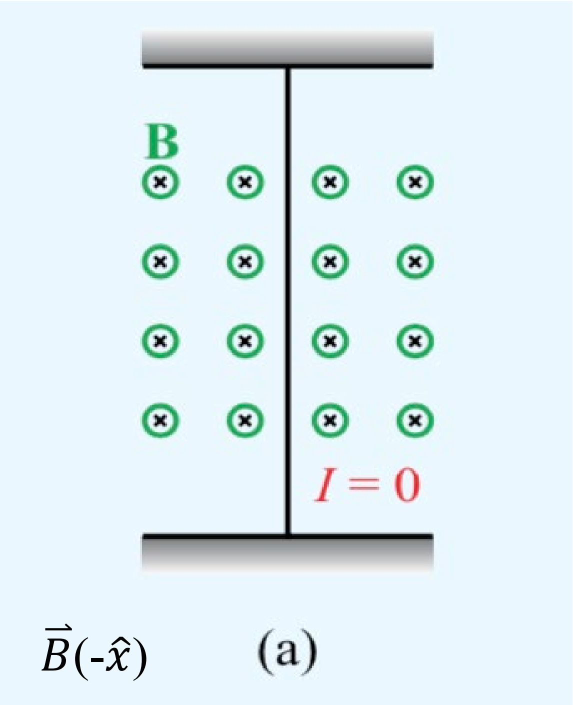
$\perp$ to $\vec{B}$; acts only on moving charge

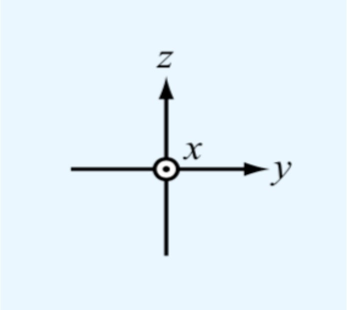


(b)

Magnetic force on current carrying conductor

Consider a conducting wire in a constant $\vec{B}$ pointing in the -$\hat{x}$ direction (into the page):

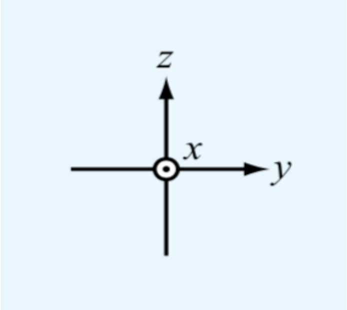


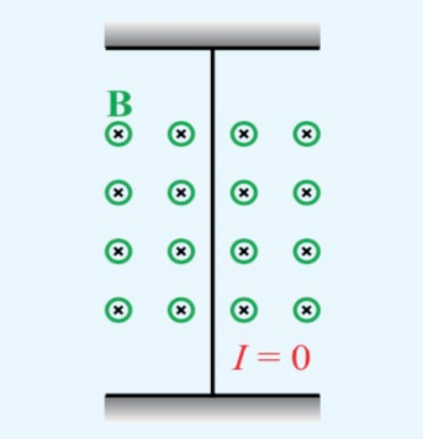


$$d\vec{F}_m = Id\vec{l} \times \vec{B} \text{ [N]}$$

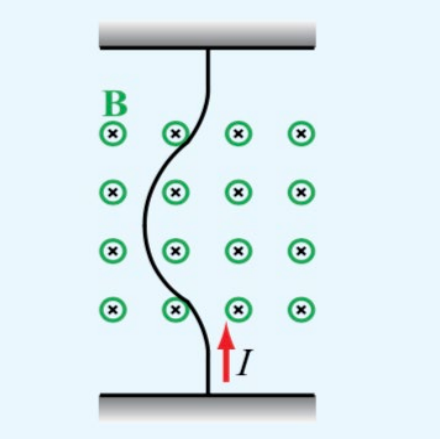
Magnetic force on current carrying conductor

Suppose the conductor now carries a current in the $+\hat{z}$





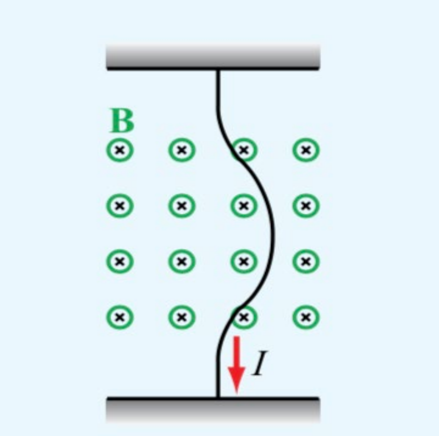
(a) $\vec{B}(-\hat{x})$



(b) $\vec{B}(-\hat{x})$

$$d\vec{F}_m = Id\vec{l} \times \vec{B} \text{ [N]}$$

What if the current is in the $-\hat{z}$



(c) $\vec{B}(-\hat{x})$

Magnetic force on closed current loop

Differential force on a differential current segment, *Idl*:

$$d\vec{F}_m = I d\vec{l} \times \vec{B}$$

For closed circuit, contour C carrying current I:

$$\vec{F}_m = I \oint_C d\vec{l} \times \vec{B}$$

If $\vec{B}$ is <u>uniform</u>, can take outside integral:

$$\vec{F}_m = I \underbrace{\left(\oint_C d\vec{l} \right)}_{= 0} \times \vec{B} = 0$$

Magnetic force on <u>closed</u> current loop in <u>uniform</u> magnetic field = 0

Magnetic force on a wire segment

Magnetostatics → Currents must flow through closed paths

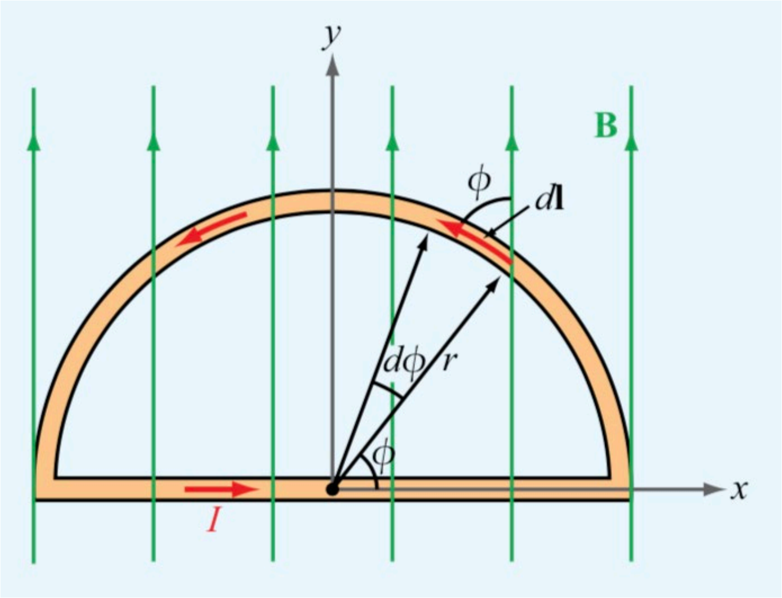
Cannot have segregated wire segment
→ must be part of loop and total $\vec{F}_m$ on loop = 0

In magnetostatics all currents flow in closed paths
→ otherwise accumulated charges at wire ends and $\frac{\partial}{\partial t} \neq 0$

We can consider magnetic force on a wire segment of a closed current loop

Wire segment: $$\vec{F}_m = I\left(\int_l d\vec{l}\right) \times \vec{B} = I\vec{l} \times \vec{B}$$

Example: magnetic force on semi-circular conductor



Consider semi circular wire, carrying current, I in x-y plane.

Uniform $\vec{B} = \hat{y}B_0$

Determine force on:
straight and circular sections

$\vec{B} = \hat{y}B_0 \qquad \vec{F}_m = I\vec{l} \times \vec{B}$

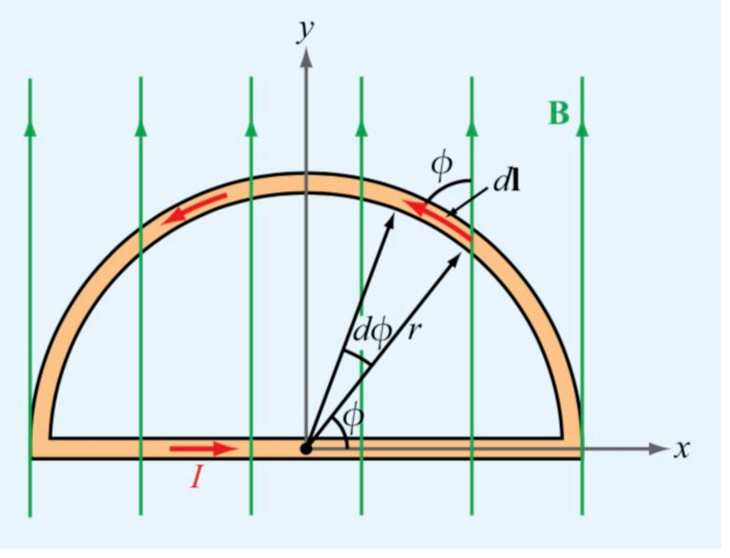
$\vec{F}_1$ on straight section: length is $2r$ along x-axis

I flows in $+x$ direction so: $\vec{l} = \hat{x}(2r)$

$$\vec{F}_1 = I\vec{l} \times \vec{B} = I\hat{x}(2r) \times \hat{y}B_0 = \hat{z}2IrB_0$$

Example: magnetic force on semi-circular conductor

Consider semi circular wire, carrying I in x-y plane. Uniform $\vec{B} = \hat{y}B_0$



$\vec{F}_2$ on semicircle section:

$d\vec{l}$ is chosen in the direction of the current: $d\vec{l} = \hat{\varphi} r d\varphi$

$$d\vec{l} \times \vec{B} \rightarrow -\hat{z}(\sin\varphi)B_0 r d\varphi$$

Angle between $d\vec{l}$ and $\vec{B}$

$$\vec{F}_2 = I\int_{\varphi=0}^{\pi} d\vec{l} \times \vec{B} = -\hat{z}I\int_{\varphi=0}^{\pi} rB_0 \sin\varphi d\varphi = -\hat{z}2IrB_0$$

$$\vec{F}_2 = -\vec{F}_1 \qquad \vec{F}_2 + \vec{F}_1 = 0$$

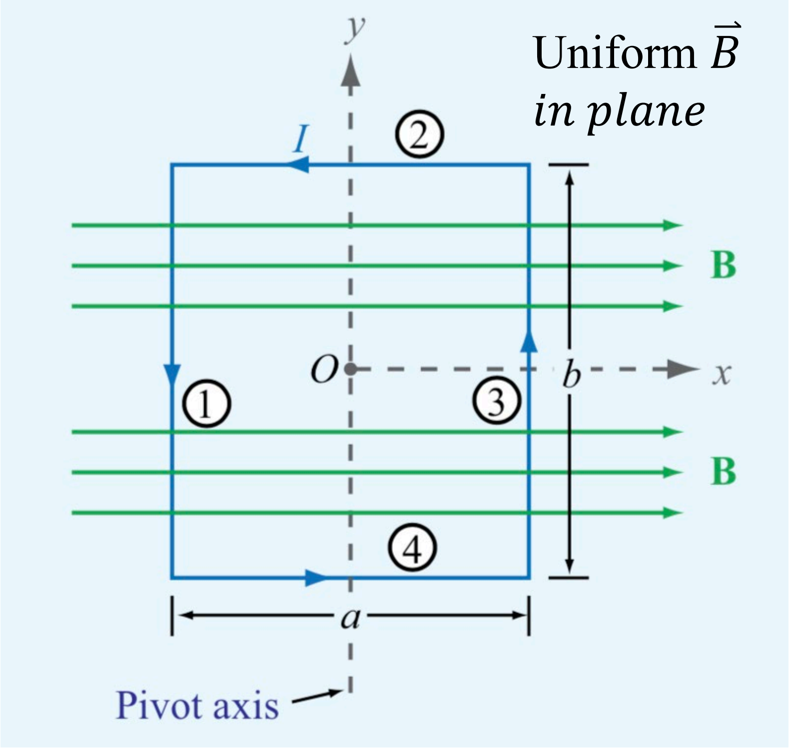
Even though total $\vec{F} = 0$ if object could rotate, we can define <u>torque</u>: $\vec{T} = \vec{d} \times \vec{F}$

Magnetic torque due to $\vec{F}_m$

Moment arm

Magnetic torque on current carrying loop

Consider loop in x-y plane, $\vec{B} = \hat{x}B_0$



Force on each of the 4 wire segments:

$$\vec{F}_1 = I(-\hat{y}b) \times \hat{x}(B_0) = \hat{z}IbB_0$$

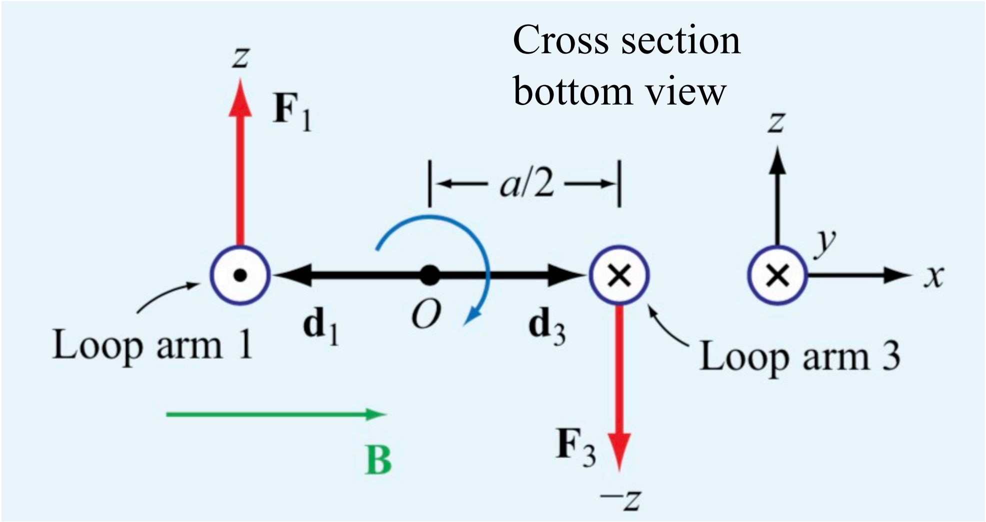
$$\vec{F}_3 = I(\hat{y}b) \times \hat{x}(B_0) = -\hat{z}IbB_0$$

$$\vec{F}_2 = 0 \qquad \vec{F}_4 = 0$$

$\longrightarrow$ $\vec{B}$ is parallel to I

Magnetic torque on current carrying loop

Consider loop in x-y plane, $\vec{B} = \hat{x}B_0$



Total force is 0 but -
$\vec{F}_1$ and $\vec{F}_3$ produce a torque
loop will rotate clockwise

Moment arm → *a*/2 $\quad \vec{d}_1 = -\hat{x}\frac{a}{2} \qquad \vec{d}_3 = +\hat{x}\frac{a}{2}$

$$\vec{T} = \vec{d}_1 \times \vec{F}_1 + \vec{d}_3 \times \vec{F}_3$$

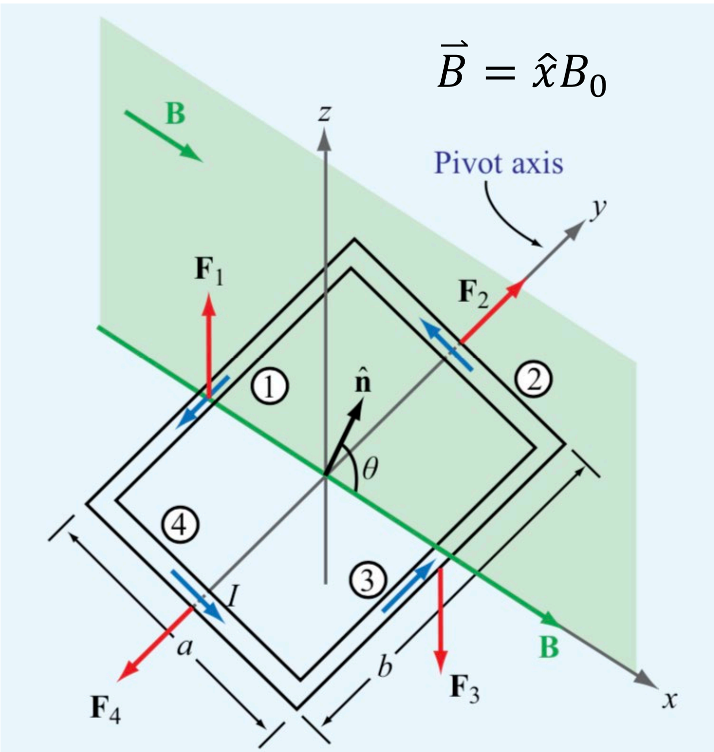
$$\vec{T} = (-\hat{x}\frac{a}{2}) \times (\hat{z}IbB_0) + (\hat{x}\frac{a}{2}) \times (-\hat{z}IbB_0)$$

$$\vec{T} = \hat{y}IabB_0 = \hat{y}IAB_0 \qquad A = ab \;\rightarrow \textit{loop cross sectional area}$$

Rectangular Loop

Now the loop rotates

but after ¼ turn: Torque → 0



$\vec{F}_2$ and $\vec{F}_4$ still along rotation axis

Only $\vec{F}_1$ and $\vec{F}_3$ contribute to torque

$\hat{n}$ = surface normal to loop

RHR: fingers around direction of current and thumb along direction of $\hat{n}$

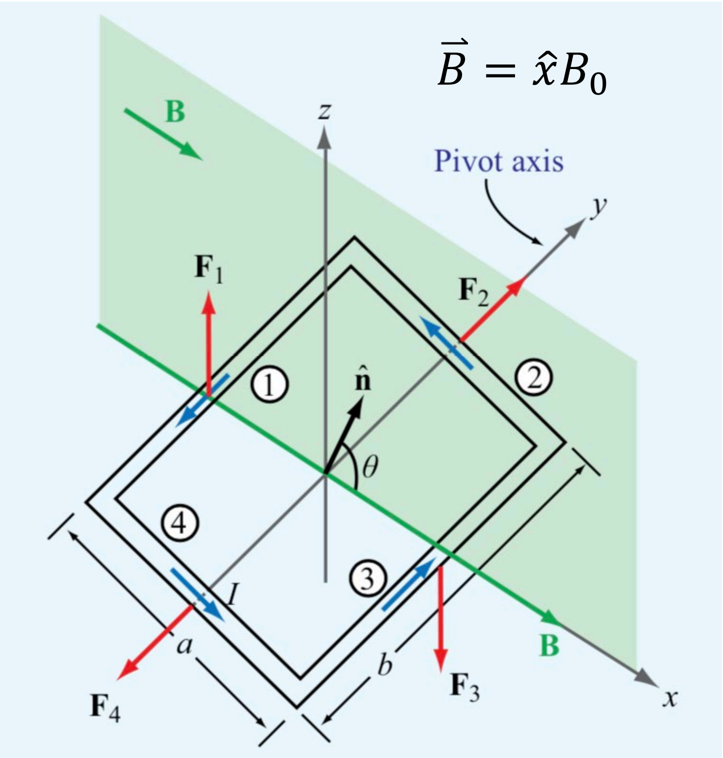
<u>moment arm</u> = $(\frac{a}{2})\sin\theta$

Net torque: $T = IAB_0\sin\theta$ $\quad\vec{B} = \hat{x}B_0$

$\theta = 90^\circ \rightarrow \vec{B}$ is parallel to loop; Torque max

$\theta = 0^\circ \rightarrow \vec{B}$ is perpendicular to loop; Torque = 0

Rectangular Loop



moment arm = $(\frac{a}{2})\sin\theta$ $\quad \vec{B} = \hat{x}B_0$

Net torque: $T = IAB_0 \sin\theta$

$\theta = 90^\circ \rightarrow \vec{B}$ is parallel to loop; Torque max

$\theta = 0^\circ \rightarrow \vec{B}$ is perpendicular to loop; Torque = 0

If we add N turns of wire:

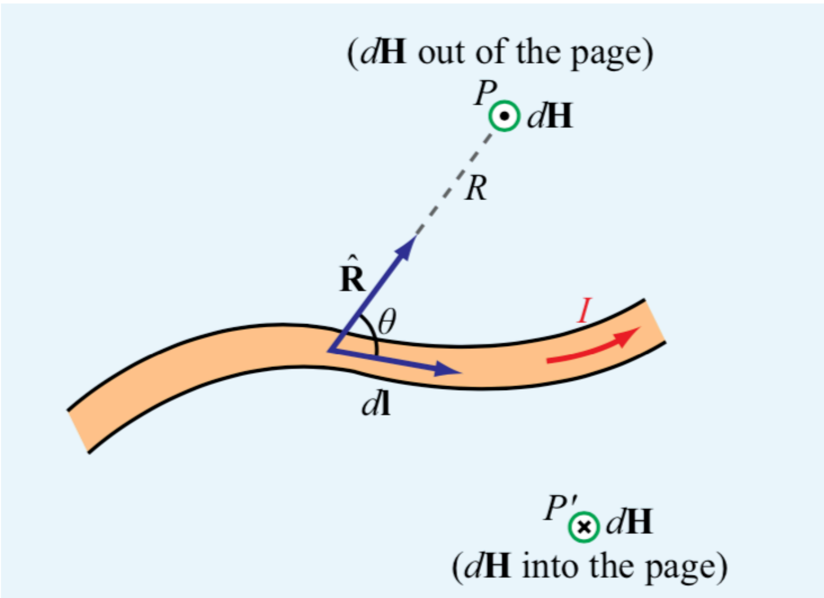
$$T = \underbrace{NIAB_0}_{\text{"magnetic moment" magnitude}} \sin\theta$$

$$\vec{m} = \hat{n}NIA = \hat{n}\mathrm{m}$$

$\vec{T} = \vec{m} \times \vec{B}$ [N ·m] And applies to any shape, any direction $\vec{B}$

Biot-Savart Law

Jean Biot + Felix Savart → relate magnetic field $\vec{H}$ to its generating current, $\vec{I}$



Magnetic field, $d\vec{H}$ generated by current element $Id\vec{l}$

$d\vec{l}$: along direction of current flow

$\hat{R}$: points <u>from</u> current element <u>to</u> observation point

$d\vec{H}$ varies with: $\frac{1}{R^2}$ (similar to $\vec{E}$)

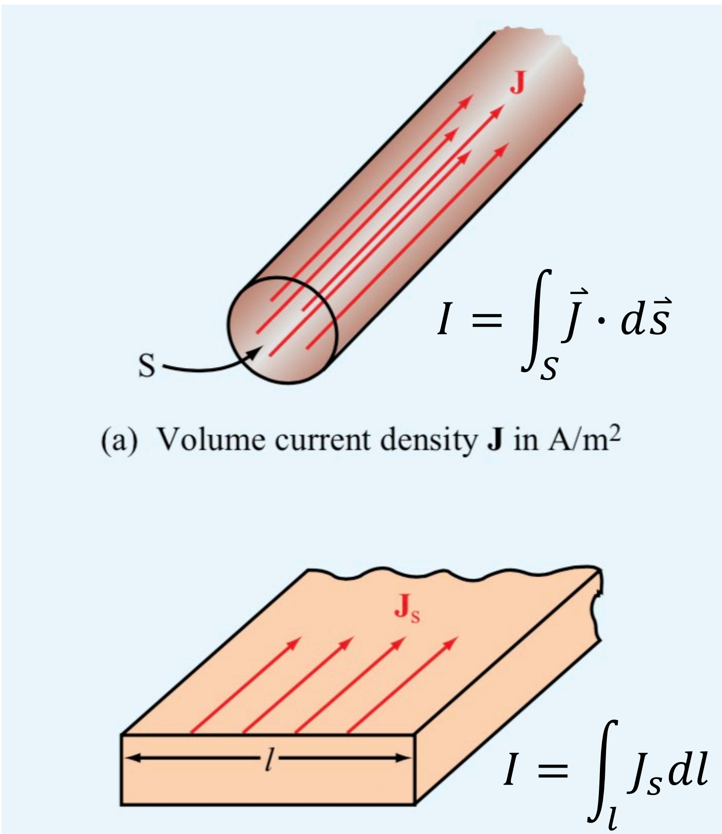
Note:
$\vec{E}$ points along $\vec{R}$
$\vec{H}$ is <u>orthogonal</u> to $d\vec{l}$ and $\hat{R}$

$$d\vec{H} = \frac{I}{4\pi}\frac{d\vec{l} \times \hat{R}}{R^2} \quad \text{[A/m]} \qquad (\vec{R} = \hat{R}R)$$

$d\vec{H}$ generated by steady current, I flowing through $d\vec{l}$

Magnetic field due to current and current distributions

To determine total magnetic field, $\vec{H}$ → sum up contributions of current elements:



(a) Volume current density **J** in A/m^2

(b) Surface current density $\mathbf{J}_s$ in A/m

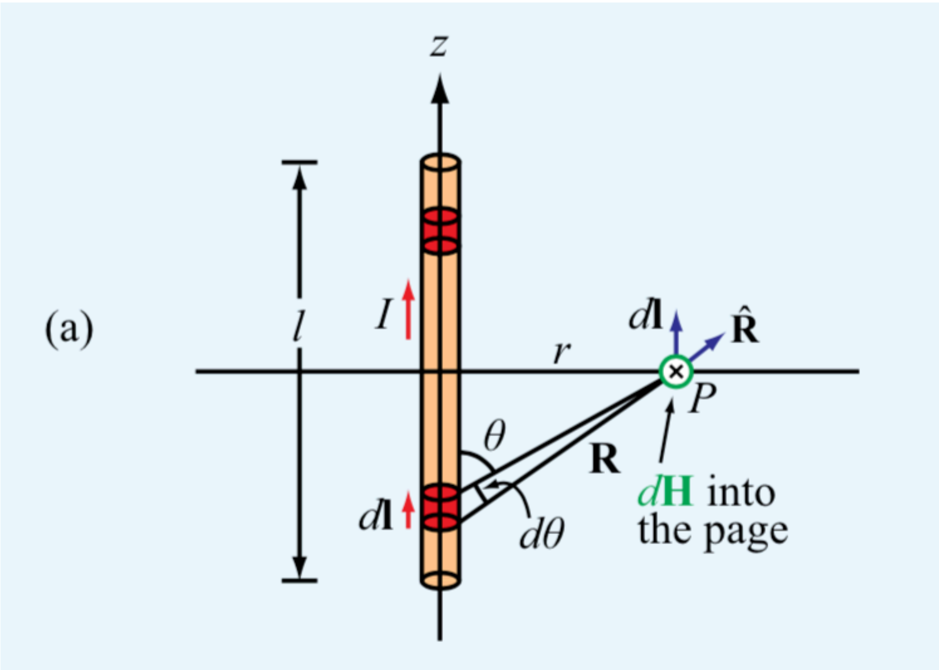
$$\boxed{\vec{H} = \frac{I}{4\pi} \int_l \frac{d\vec{l} \times \hat{R}}{R^2}}$$

$$I d\vec{l} \rightarrow \vec{J}_s ds \rightarrow \vec{J} d\mathcal{V} \quad \text{(volume)}$$

$$\vec{H} = \frac{I}{4\pi} \int_{\mathcal{V}} \frac{\vec{J} \times \hat{R}}{R^2} d\mathcal{V}$$ Volume current

$$\vec{H} = \frac{I}{4\pi} \int_S \frac{\vec{J}_s \times \hat{R}}{R^2} ds$$ Surface current

Magnetic field of linear conductor



Conductor length, l carrying current I along $+\hat{z}$

Determine $\vec{B}$ at point, P, located distance r in the x-y plane

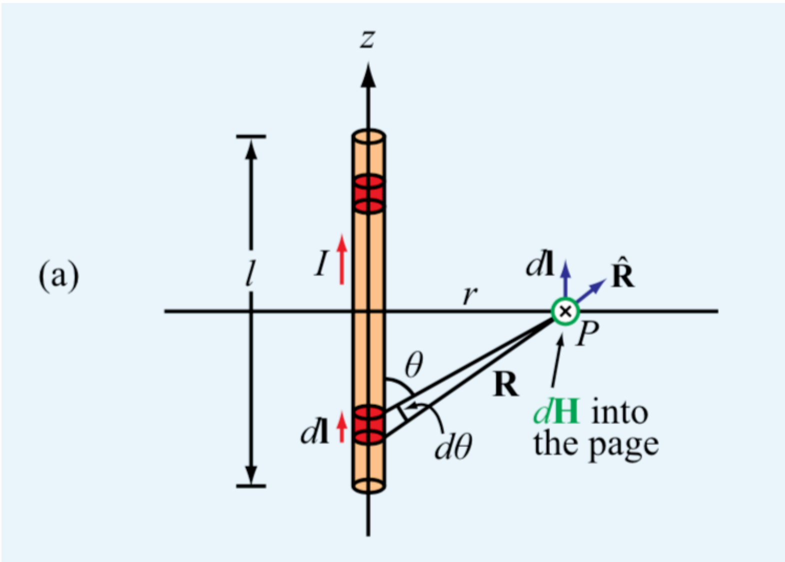
$$\vec{H} = \frac{I}{4\pi}\int_l \frac{d\vec{l} \times \hat{R}}{R^2}$$

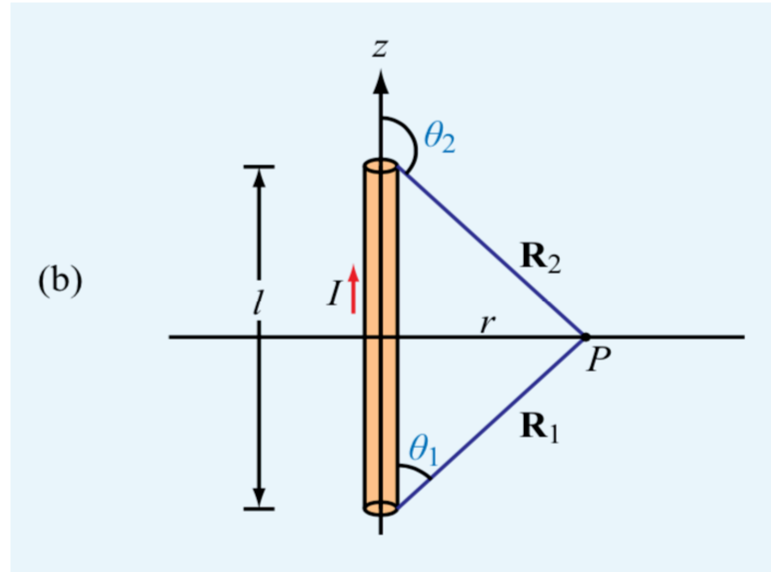
Differential length vector: $d\vec{l} = \hat{z}dz$

$$d\vec{l} \times \hat{R} = dz(\hat{z} \times \hat{R}) = \underbrace{\hat{\varphi} sin\theta dz}$$

θ = angle between $d\vec{l}$ and $\hat{R}$

Magnetic field of linear conductor





Limiting angles θ_1, θ_2 each measured between vector $Id\vec{l}$ and vector connecting end of conductor associated with that angle to point *P*

$$\vec{H} = \frac{I}{4\pi}\int_{z=-l/2}^{z=l/2} \frac{d\vec{l} \times \hat{R}}{R^2} = \hat{\varphi}\,\frac{I}{4\pi}\int_{-l/2}^{l/2} \frac{sin\theta}{R^2}\,dz$$

Both *R* and θ depend on *z*

Magnetic field of linear conductor

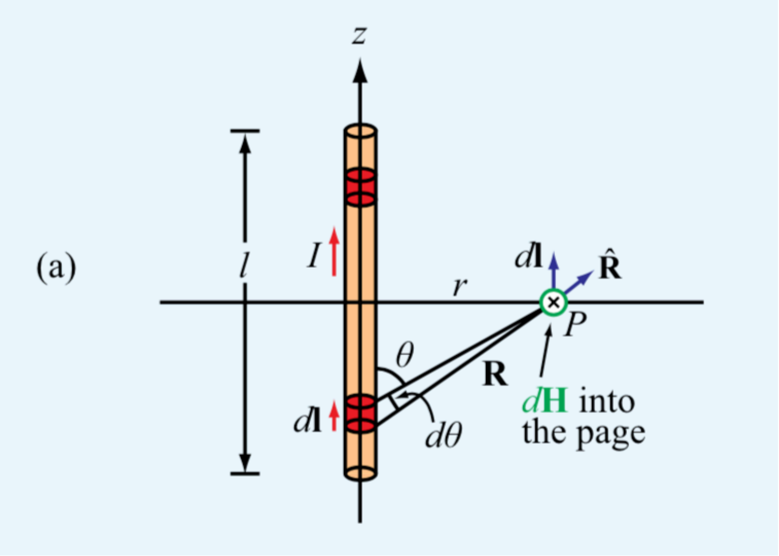
$$\vec{H} = \hat{\varphi}\frac{I}{4\pi}\int_{-l/2}^{l/2}\frac{sin\theta}{R^2}dz$$ both R and θ depend on z

$$csc\theta = \frac{1}{sin\theta}$$

We change integration variable from z → to → θ

$$tan\theta = \frac{-r}{z}$$

$$R = rcsc\theta \qquad z = -rcot\theta \qquad dz = rcsc^2\theta d\theta \qquad sin\theta = \frac{r}{R}$$



$$\vec{H} = \hat{\varphi}\frac{I}{4\pi}\int_{\theta_1}^{\theta_2}\frac{sin\theta rcsc^2\theta d\theta}{r^2csc^2\theta}$$

$$\vec{H} = \hat{\varphi}\frac{I}{4\pi r}\int_{\theta_1}^{\theta_2}sin\theta d\theta$$

$$= \hat{\varphi}\frac{I}{4\pi r}(cos\theta_1 - cos\theta_2)$$

$$cos\theta_1 = \frac{l/2}{\sqrt{r^2+(l/2)^2}} \qquad cos\theta_2 = -cos\theta_1 = \frac{-l/2}{\sqrt{r^2+(l/2)^2}}$$

Magnetic field of linear conductor

$$\vec{B} = \mu_0 \vec{H} = \hat{\varphi} \frac{\mu_0 I l}{2\pi r \sqrt{4r^2 + l^2}}$$

For infinite long wire, $l \gg r \rightarrow \quad \vec{B} = \hat{\varphi} \frac{\mu_0 I}{2\pi r}$

$\vec{B}$ forms concentric circles around wire, with magnitude $\sim \frac{1}{r}$